Concordance Lecture 10

☐ Mic check ☐
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LAST TIME: Khovanov homology

link cobordisms $S: L_0 \to L_1$ $\rightsquigarrow$ $\mathbb{Z}$-linear maps $Kh(S): Kh(L_0) \longrightarrow Kh(L_1)$

NOTE Maps from last time encode χ in q-grading


$\chi = -1$ induces $d:)\cdot(\longrightarrow \asymp$ with $(h+1, q-1)$ bigrading


$\chi = 1$ induces $\iota: \varnothing \longrightarrow \bigcirc$ (unknot) with $(h, q+1)$ bigrading

Maps encode χ ☺

① LEE HOMOLOGY (E.S. Lee 02)

Define $\mathcal{C}'(D) = \langle \text{labeled smoothings} \rangle_{\mathbb{Q}}$

Define new differential d' with same construction, but new maps m' and Δ'

$$m' \begin{cases} +)\,(+ \longmapsto \overset{+}{\asymp} \\ +)\,(- \longmapsto \overset{-}{\asymp} \\ -)\,(+ \longmapsto \overset{-}{\asymp} \\ -)\,(- \longmapsto \overset{+}{\asymp} \end{cases} \qquad \Delta' \begin{cases})+(\longmapsto \overset{+}{\underset{-}{\asymp}} + \overset{-}{\underset{+}{\asymp}} \\ )-(\longmapsto \overset{-}{\underset{-}{\asymp}} + \overset{+}{\underset{+}{\asymp}} \end{cases}$$

DEFN The Lee chain cx is the pair $(\mathcal{C}'(D), d')$ with assoc. Lee homology groups $Lee(D)$.

SIMILAR
- Also bigraded by h and q
- Also a link invariant

DIFFERENT

Ⓐ d' does not respect q-grading (not homogeneous)

$$\underset{h,q}{-)\cdot(} \longmapsto \underset{h+1,q}{\overset{-}{\underset{-}{\asymp}}} + \underset{h+1,q+4}{\overset{+}{\underset{+}{\asymp}}}$$

BUT q-grading always increases!

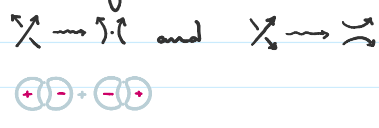
↳ leads to spectral sequence $\underset{E_1}{Kh(D)} \rightrightarrows \underset{E_\infty}{Lee(D)}$

(B) For a link cobordism $S: L_0 \to L_1$, there is an induced map $\mathrm{Lee}(S): \mathrm{Lee}(K_0) \longrightarrow \mathrm{Lee}(K_1)$
"filtered with degree $\chi(S)$" i.e. $q(x) \leq q(\mathrm{Lee}(S)(x)) - \chi(S)$ $\forall x \in \mathrm{Lee}(L_0)$

Previously had equality

(C) Thm (Lee) $\mathrm{Lee}(L) \cong \mathbb{Q}^{2^m}$ where $m = \#$ components in L.

Proof idea Build bijection $\{\text{generators}\} \longrightarrow \{\text{orientations of } L\}$
- Choose an orientation θ for diagram D of L
- Produce smoothing σ with θ, i.e.

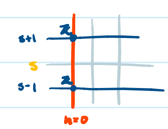


- Orientation of components gives label ℓ (not important)
$\mathfrak{s}_\theta := (\sigma, \ell)$ is a gen for $\mathrm{Lee}(D)$

(2) Lee Homology of Knots (J. Rasmussen 04)

$\mathrm{Lee}(\text{Knot}) \cong \mathbb{Z} \oplus \mathbb{Z}$, but in which bigradings?

Thm 0 (Ras) For a knot K, $\exists s \in 2\mathbb{Z}$ s.t. $\mathrm{Lee}(K) = \begin{cases} \mathbb{Q} & h=0,\ q = s \pm 1 \\ 0 & \text{otherwise} \end{cases}$
with generators $\mathfrak{s}_\theta$ and $\mathfrak{s}_{\bar{\theta}}$ ← reverse orientation (not mirror!)



DEFN The s-invariant of a knot K is $s(K) := s \in 2\mathbb{Z}$

FACT For links, bigrading of $Kh(L) \cong \mathbb{Q}^{2^m}$ is given by linking number

Thm 1 (Ras) $|s(K)| \leq 2g_4(K)$ g_4 = smooth 4 genus

Proof in a second

Cor K sm. slice $\Rightarrow s(K) = 0$

Ex $s(\mathcal{U}) = 0$
Ex $s(3_1) = 2$
Ex $K = P(-3,5,7)$ has $s(K) \neq 0$ and $\Delta_K = 1$

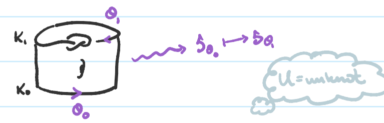
-3 5 7

FACTS (1) Similar to σ

$s(K_0 \# K_1) = s(K_0) + s(K_1)$
$s(\overline{K}) = -s(K)$
$\Rightarrow$ s is a concordance inv't (HW)
$\Rightarrow$ $[3_1]$ has ∞-order in $\mathcal{C}_{sm}$

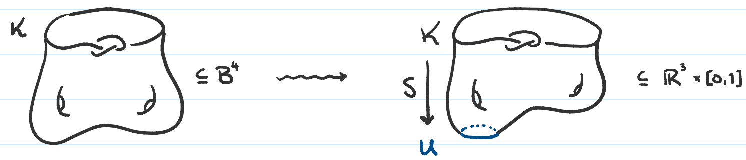
(2) $\exists$ knots with $|s(K)| > \sigma(K)$ and others with $<$

Thm 1
Proof Sketch



SET UP Lemma: If $S: K_0 \to K_1$ is connected, $Lee(S)$ is an iso

SET UP If K bounds a surface of genus g in B^4, then $\exists$ link cob. $S: K \to \mathcal{U}$ with $\chi(S) = -2g$



PROOF Want to compute s-invariant so analyze $x \in Lee(K)$ with $q(x) = s(K) + 1$

$Lee(S)$ an iso, so can also analyze $y = Lee(S)(x) \neq 0$

$\hookrightarrow s(\mathcal{U}) = 0$ so $q(y) \leq 1$

$q(x)$ and $q(y)$ related by filtered degree of $Lee(S)$ (ie (B) above)

$\hookrightarrow q(y) - \chi(S) \geq q(x) = s(K) + 1$

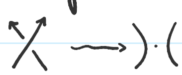
Together: $1 - \chi(S) \geq s(K) + 1$ or $2g \geq s(K)$

Repeat argument with $\overline{S}: \mathcal{U} \to \overline{K}$ and $s(\overline{K}) = -s(K)$ to get
$s(K) \geq -2g$

$\square$

Thm 2 If K is a positive knot, $s(K) = g_3(K) = g_4(K)$

Proof idea Consider $\alpha_0 :=$ 0 induced smoothing with all v_- label
All smoothings are 0-smoothings



Let k = # components in smoothing
α_0 represents non-triv. Lee class (why?)
Must have $s(K) - 1 = q(\alpha_0)$ since no class can be lower in $Lee^{0,q}(K)$

$\hookrightarrow$ recall $q(\alpha) = v_+ - v_- + h + wr(D)$
$\uparrow$ only way lower is with more v_-'s but not possible; $\uparrow$ 0; $\uparrow$ fixed

So $s(K) = -(\# v_-) + n + 1$

Also, Seifert's algorithm gives surface of genus $\frac{k-n+1}{2}$ (why?)

So $g_3(K) \leq \frac{k-n+1}{2}$

Thus, $g_3(K) \leq \frac{|k-n+1|}{2} = \frac{|s(K)|}{2} \leq g_4(K) \leq g_3(K)$ $\square$

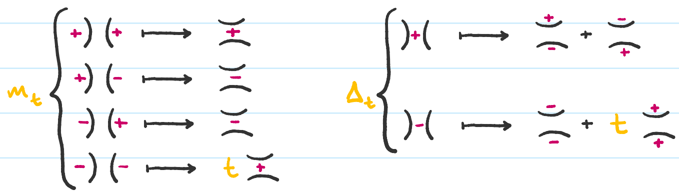
Thm 3 If knots $K_\pm$ differ by a crossing, from pos in K_+ to neg. in K_- then

$$s(K_-) \leqslant s(K_+) \leqslant s(K_-)+1$$

Can be used to reprove SBI

③ GENERALIZATIONS

Let t be an indeterminant and set



Follow same process as before to define $(\mathcal{C}_t, d_t)$ and Kh_t

$Kh_0 = Kh$

$Kh_1 = $ Lee

Kh_t often called "Lee homology"

There are many similar/different variants

- universal Kh (w/ variable h)
- Bar-Natan homology

  } "deformations of Kh"
- reduced homology (want $Kh(\text{unknot}) = \mathbb{Z}$ but $= \mathbb{Z} \oplus \mathbb{Z}$)
- tangle homology
- odd Kh
- Khovanov-Rozansky sl_N

Our Kh is even, unreduced, undeformed, sl_2 homology

Lee is a deformation